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ADDENDUM

A remark on invariant symmetric second rank spinors and electromagnetic tensors

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Abstract. Invariance conditions on symmetric second rank spinors lead to the determination of their non-equivalent kinematical groups. Connections with electromagnetic tensors are discussed through Carmeli's classification.

In this journal we have recently studied (Beckers and Hussin 1981) *invariance arguments* connecting *mixed*, second-rank, Hermitian spinors and real four-vectors. This analysis illustrates and enhances correspondences between spinors and tensors: the above case corresponds to the simplest one, i.e. between a mixed '(1, 1)-spinor' and a '1-index tensor' or four-vector (Misner *et al* 1973)

$$\chi_{\dot{U}A} = \sigma^{\mu}{}_{\dot{U}A} A_{\mu} \tag{1}$$

where the $\sigma^{\mu}{}_{\dot{U}A}$ are the Infeld–Van der Waerden symbols (Infeld and Van der Waerden 1933, Bade and Jehle 1953). Such a study is of special interest if we recall that the four-vector A can be seen as a four-potential (when Maxwell theory is under consideration).

Here we want to emphasise another physically interesting correspondence between spinors and tensors through invariance arguments. It is well known that *skew-symmetric* second-rank tensors F and *symmetric* second-rank spinors η are in correspondence (Pirani 1964, Carmeli 1977). We have

$$F^{\mu\nu} = \sigma^{\mu}{}_{\dot{U}A}\sigma^{\nu}{}_{\dot{V}B}(C^{\dot{U}\dot{V}}\eta^{AB} + C^{AB}\eta^{\dot{U}\dot{V}}) \tag{2}$$

where

$$C = (C^{AB}) = (C^{\dot{U}\dot{V}}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3}$$

is the metric spinor. The physical interest lies in the meaning of the tensor F which can be seen as the electromagnetic tensor (when Maxwell theory is under consideration).

Under the Poincaré group such a spinor η transforms in the following usual way

$$\eta'^{AB}(x') = \eta'^{AB}[(1+\omega)x + \alpha] = L^{A}{}_{C}L^{B}{}_{D}\eta^{CD}(x) \tag{4}$$

where the infinitesimal form of the matrix L can be written

$$L = \mathbb{1} - \tfrac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}) \tag{5}$$

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with

$$\mathbf{\Omega} = \{\Omega^k = \phi^k + \mathrm{i}\theta^k\}, \qquad \omega = (\boldsymbol{\phi}, \boldsymbol{\theta}) \tag{6}$$

and where the $\boldsymbol{\sigma}$ are the well known 2×2 Pauli matrices. Following the method developed by Beckers and Hussin (1981), § 3, we obtain the following *invariance conditions* on η

$$(\mathbf{\Omega}\cdot\boldsymbol{\sigma})^A{}_C\eta^{CB}(x) + (\mathbf{\Omega}\cdot\boldsymbol{\sigma})^B{}_E\eta^{AE}(x) = 2D\eta^{AB}(x) \tag{7}$$

where D is simply

$$D \equiv (\boldsymbol{r}\cdot\boldsymbol{\phi})\partial/\partial t + (t\boldsymbol{\phi} + \boldsymbol{r}\Lambda\boldsymbol{\theta})\cdot\partial/\partial\boldsymbol{r} - \alpha\cdot\nabla. \tag{8}$$

Explicitly we obtain:

$$\begin{aligned} &(\Omega^3 - D)\eta^{11} + (\Omega^1 - \mathrm{i}\Omega^2)\eta^{12} = 0,\\ &(\Omega^1 + \mathrm{i}\Omega^2)\eta^{11} - 2D\eta^{12} + (\Omega^1 - \mathrm{i}\Omega^2)\eta^{22} = 0,\\ &(\Omega^1 + \mathrm{i}\Omega^2)\eta^{12} - (\Omega^3 + D)\eta^{22} = 0. \end{aligned} \tag{9}$$

The kinematical groups of a *constant and uniform* spinor $\eta\,(D\eta = 0)$ can then be easily determined by using its form in terms of the basic symmetric spinors (Carmeli 1977) η_0, η_1, η_2 defined by

$$\eta_0 = (\eta_{0AB}) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \qquad \eta_1 = (\eta_{1AB}) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \eta_2 = (\eta_{2AB}) = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{10}$$

If $\eta = \eta_0$, the invariance conditions (9) become

$$\left.\begin{aligned}&\Omega^3 = 0\\ &\Omega^1 + \mathrm{i}\Omega^2 = 0\end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned}&\phi^3 = \theta^3 = 0\\ &\phi^1 - \theta^2 = \phi^2 + \theta^1 = 0\end{aligned}\right. \tag{11}$$

so that the corresponding kinematical group is

$$\mathrm{G}_0 \equiv \{P^\mu, A^1 = K^1 + J^2, A^2 = K^2 - J^1\} \tag{12}$$

where the $\boldsymbol{J}$ and $\boldsymbol{K}$ generators are associated with pure Lorentz rotations and boosts respectively.

If $\eta = \eta_1$, equations (9) give

$$\left.\begin{aligned}&\Omega^1 + \mathrm{i}\Omega^2 = 0\\ &\Omega^1 - \mathrm{i}\Omega^2 = 0\end{aligned}\right\} \Longleftrightarrow \phi^1 = \phi^2 = \theta^1 = \theta^2 = 0 \tag{13}$$

and the kinematical group is simply

$$\mathrm{G}_1 \equiv \{P^\mu, J^3, K^3\}. \tag{14}$$

If $\eta = \eta_2$, we obtain

$$\left.\begin{aligned}&\Omega^1 - \mathrm{i}\Omega^2 = 0\\ &\Omega^3 = 0\end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned}&\phi^1 + \theta^2 = \phi^2 - \theta^1 = 0\\ &\phi^3 = \theta^3 = 0\end{aligned}\right. \tag{15}$$

corresponding to the kinematical group

$$\mathrm{G}_2 \equiv \{P^\mu, A'^1 = K^1 - J^2, A'^2 = K^2 + J^1\} \tag{16}$$

which is conjugated to G_0: the Lie algebras of G_0 and G_2 are isomorphic to each other.

So, there exist *two* non-equivalent structures for the kinematical groups of a constant and uniform spinor η. These results correspond to those of Bacry *et al* (1970) on constant and uniform electromagnetic tensors F as expected from equation (2).

Let us now complete the remark in connecting these results with specific *physical* cases. *First*, let us notice that Carmeli (1977) gave a classification and the canonical forms of symmetric spinors η with respect to the values of their η determinant:

— if $\det \eta = 0$, the canonical form is

$$\eta_C = \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \eta'_C = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon' \end{pmatrix} \tag{17}$$

where ε, ε' are arbitrary complex numbers;

— if $\det \eta \neq 0$, the canonical form is

$$\eta''_C = \begin{pmatrix} 0 & \varepsilon'' \\ \varepsilon'' & 0 \end{pmatrix}. \tag{18}$$

Secondly, let us take into account the two types of electromagnetic tensors $F \equiv (\boldsymbol{E}, \boldsymbol{B})$ given by Bacry *et al* (1970):

— if $\boldsymbol{E}$ and $\boldsymbol{B}$ are *orthogonal*, $|\boldsymbol{E}| = |\boldsymbol{B}|$, (for example $\boldsymbol{E} = (E, 0, 0)$, $\boldsymbol{B} = (0, E, 0)$)— consequently F admits G_0 as a kinematical group—we get from Carmeli's considerations the associated spinor

$$\eta_C = \begin{pmatrix} -E & 0 \\ 0 & 0 \end{pmatrix} \tag{19}$$

in correspondence with equation (17);

— if $\boldsymbol{E}$ and $\boldsymbol{B}$ are *parallel* (along the z axis, for example: $\boldsymbol{E} = (0, 0, E)$, $\boldsymbol{B} = (0, 0, B)$) consequently F admits G_1 as a kinematical group, we also get

$$\eta''_C = \begin{pmatrix} 0 & -\frac{1}{2}(E - \mathrm{i}B) \\ -\frac{1}{2}(E - \mathrm{i}B) & 0 \end{pmatrix} \tag{20}$$

in correspondence with equation (18).

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